Stochastic calculus on Wasserstein spaces

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This talk is based on a joint work with Hao Ding and Xiangdong Li.

I will not give a survey on the development of the whole topic, apart from well-known pioneer works by Y. Brenier, R. McCann, F. Otto, J. Lott, K.T. Sturm, L. Ambrosio, I only cite here the following papers from which I've learnt a lot.

• Songzi Li and Xiang-dong Li, W-entropy formulas and Langevin deformation of flows on the Wasserstein space over Riemannian manifolds, arXiv:1604.02596v1.

• Wei Liu, Liming Wu and Chaoen Zhang, Long-time behaviors of mean-field interacting particle systems related to McKean-Vlasov equations. *Comm. Math. Phys.* 387 (2021), 179-214.

• Feng-Yu Wang, Diffusions and PDEs on Wasserstein Spaces, *arXive: 1903.02148v2*, 2019.

Framework

Let M be a connected compact Riemannian manifold, of the distance d_M, with the measure dx such that $\int_M dx = 1$. As usual, we denote by $\mathbb{P}_2(M)$ the space of probability measures on M, endowed with the Wasserstein distance W_2 defined by

$$W_2^2(\mu_1, \mu_2) = \inf\left\{\int_{M\times M} d_M^2(x, y)\, \pi(dx, dy), \quad \pi \in C(\mu_1, \mu_2)\right\},$$

where $C(\mu_1, \mu_2)$ is the set of probability measures π on $M \times M$, having μ_1, μ_2 as two marginal laws. It is well known that $\mathbb{P}_2(M)$ endowed with W_2 is a compact space. In this work, we will be concerned with the subspace $\mathbb{P}_{2,\infty}(M)$ of measures having positive smooth density.

For the tangent space $\bar{\mathbf{T}}_\mu$ of $\mathbb{P}_2(M)$ at μ, we adopt the definition given by L. Ambrosio and all, that is,

$$\bar{\mathbf{T}}_\mu = \overline{\{\nabla\psi,\ \psi \in C^\infty(M)\}}^{L^2(\mu)},$$

the closure of gradients of smooth functions in the space $L^2(\mu)$ of vector fields on M : they used absolutely continuous curves to unify different types of curves in $\mathbb{P}_2(M)$.

A curve $\{c(t);\ t \in [0,1]\}$ in $\mathbb{P}_2(M)$ is said to be absolutely continuous if there exists $k \in L^2([0,1])$ such that

$$W_2\big(c(t_1), c(t_2)\big) \leq \int_{t_1}^{t_2} k(s)\, ds, \quad t_1 < t_2.$$

For such a curve, there exists a Borel vector field Z_t on M in $L^2([0,1] \times M)$ such that the continuity equation holds

$$\frac{dc_t}{dt} + \nabla \cdot (Z_t c_t) = 0.$$

The uniqueness of solutions to above equation holds if $Z_t \in \bar{\mathbf{T}}_{c_t}$ for almost all $t \in [0,1]$. We say that Z_t is the intrinsic derivative of $\{c_t\}$ and denote it by

$$\frac{d^I c_t}{dt}.$$

We will also use constant vector fields V_ψ on $\mathbb{P}_2(M)$. More precisely, for $\psi \in C^\infty(M)$, we consider the ODE

$$\frac{dU_t}{dt} = \nabla\psi(U_t), \quad U_0(x) = x,$$

and let $c_t = (U_t)_\# \mu$ with μ given. In this case, $\dfrac{d^I c_t}{dt} = \nabla\psi$. We say that a functional $\mathcal{F}$ is derivable along V_ψ if

$$(\bar{D}_{V_\psi}\mathcal{F})(\mu) = \Big\{\frac{d}{dt}\mathcal{F}((U_t)_{\#\mu})\Big\}_{t=0} \quad \text{exists.}$$

The gradient $\bar{\nabla}\mathcal{F}(\mu) \in \bar{\mathbf{T}}_\mu$ exists if $(\bar{D}_{V_\psi}\mathcal{F})(\mu) = \langle \bar{\nabla}\mathcal{F}, V_\psi\rangle_{\bar{\mathbf{T}}_\mu}$.

Here are usual functionals considered in literature, see for example monographs by Villani, Ambrosio and all.

1) **Potential energy functional.** $F_\varphi(\mu) = \int_M \varphi \; \mu(dx)$, for $\varphi \in C^2(M)$.

2) **Internal energy functional.** Let $\chi : [0, +\infty[\to] - \infty, +\infty]$ be a proper, continuous convex function. The internal energy $\mathcal{F}$ is defined as follows

$$\mathcal{F}(\mu) = \int_M \chi(\rho(x))\, dx, \quad \text{if} \;\; d\mu = \rho\, dx,$$

and $\mathcal{F}(\mu) = +\infty$ otherwise. Two important examples are $\chi(s) = s \log(s)$ and $\chi(s) = \dfrac{s^m}{m-1}$ for $m > 1$.

3) **Interaction energy functional.** Let $W : M^2 \to] - \infty, +\infty]$ be a l.s.c function, we define

$$\mathcal{W}(\mu) = \int_{M \times M} W(x, y)\mu(dx)\mu(dy).$$

Internal energy functional $\mathcal{F}$ plays a particular role. However for $\chi(s) = s\log(s)$,

$$\mu \to \mathcal{F}(\mu) \quad \text{is not continuous.}$$

We need the following explicit expression (see Villani or Ambrosio)

Theorem

For $\chi \in C^2(\mathbb{R}^*)$ such that $|\chi(s)| + s|\chi'(s)| + s^2|\chi''(s)|$ is bounded over $[0,1]$, and $d\mu = \rho\, dx$,

$$(\bar{D}_{V_\psi}\bar{D}_{V_\psi}\mathcal{F})(\mu) = \int_M \tilde{\chi}'(\rho)(\Delta\psi)^2\rho^2\, dx - \int_M \tilde{\chi}(\rho)\langle \nabla\psi, \nabla\Delta\psi\rangle\, \rho\, dx,$$

where $\tilde{\chi}(s) = \chi'(s) - \dfrac{\chi(s)}{s}$. And

$$(\bar{D}_{V_\psi}\mathcal{F})(\mu) = -\int_M \left(\chi'(\rho)\rho - \chi(\rho)\right)\Delta\psi\, dx$$

SDE on $\mathbb{P}_{2,\infty}(M)$

Given a family of vector fields $\{A_0(t,\cdot), A_1(t,\cdot), \dots, A_N(t,\cdot)\}$ on M, and $B_t^0 = t$ and $(B_t^1, \dots, B_t^N)$ a BM on $\mathbb{R}^N$, how to understand the following SDE on M

$$\circ\, d_t X_{t,s} = \sum_{i=0}^{N} A_i(t, X_{t,s}) \circ dB_t^i, \quad X_{s,s}(x) = x \ ?$$

The above equality formally holds in the tangent space $T_{X_{t,s}}M$. $\{X_{t,s},\ t \geq s\}$ is a solution to SDE if for any $f \in C^2(M)$, it holds

$$\begin{aligned} f(X_{t,s}) =& f(x) + \sum_{i=0}^{N} \int_s^t (\mathcal{L}_{A_i(u)} f)(X_{u,s})\, dB_u^i \\ &+ \frac{1}{2} \sum_{i=1}^{N} \int_s^t \big(\mathcal{L}_{A_i(u)}^2 f\big)(X_{u,s})\, du, \end{aligned}$$

which simply comes from: $d_t f(X_{t,s}) = \langle \nabla f(X_{t,s}), \circ d_t X_{t,s} \rangle$.

Now what happens in $\mathbb{P}_{2,\infty}(M)$? Let $\{\phi_0, \phi_1, \ldots, \phi_N\}$ be a family of functions on $[0,1] \times M$, smooth enough in $x \in M$. In this talk, ∇ always denotes the gradient operator on M. Consider the following Stratanovich SDE on M:

$$dX_{t,s} = \sum_{i=0}^{N} \nabla \phi_i(t, X_{t,s}) \circ dB_t^i, \quad t \geq s, \quad X_{s,s}(x) = x.$$

Let $d\mu = \rho\, dx$ be a probability measure on M, we set $\mu_t(\omega) = \big(X_{t,0}(\omega)\big)_{\#}\mu$. Let $\varphi \in C^2(M)$. First using Itô formula to $\varphi(X_{t,0})$, then integrating the two hand sides respect to $d\mu$, we have

$$\begin{aligned} \circ d_t\, F_\varphi(\mu_t) &= \sum_{i=0}^{N} \Big(\int_M \langle \nabla\varphi, \nabla\phi_i(t,\cdot)\rangle\ \mu_t(dx) \Big) \circ dB_t^i \\ &= \sum_{i=0}^{N} \langle V_\varphi, V_{\phi_i(t,\cdot)} \rangle_{\bar{\mathbf{T}}_{\mu_t}} \circ dB_t^i. \end{aligned}$$

We say that the intrinsic Itô stochastic differential of μ_t, denoted by $\circ d^I_t \mu_t$, admits the following expression

$$\circ d^I_t \mu_t = \sum_{i=0}^{N} V_{\phi_i(t,\cdot)} \ \circ dB^i_t.$$

Recall that $\bar{\nabla} F_\varphi = V_\varphi$; then $\circ d_t F_\varphi(\mu_t)$ can be written in the form

$$\circ d_t \, F_\varphi(\mu_t) = \langle \bar{\nabla} F_\varphi, \ \circ d^I_t \mu_t \rangle_{\bar{\mathbf{T}}_{\mu_t}},$$

symbolically read in the inner product of $\bar{\mathbf{T}}_{\mu_t}$. In Itô form:

$$d_t F_\varphi(\mu_t) = \sum_{i=0}^{N} \langle \bar{\nabla} F_\varphi, V_{\phi_i(t)} \rangle_{\bar{\mathbf{T}}_{\mu_t}} \, dB^i_t + \frac{1}{2} \sum_{i=1}^{n} (\bar{D}^2_{V_{\phi_i(t)}} F_\varphi)(\mu_t) \, dt,$$

where $\bar{D}^2_{V_{\phi_i(t)}} F_\varphi$ denotes the second order derivative.

Theorem

For any internal energy functional $\mathcal{F}$ with χ satisfying above conditions, we have

$$d_t\mathcal{F}(\mu_t) = \sum_{i=0}^{N} \langle \bar{\nabla}\mathcal{F}, V_{\phi_i(t)} \rangle_{\bar{\mathbf{T}}_{\mu_t}} \, dB_t^i + \frac{1}{2}\sum_{i=1}^{N} \big(\bar{D}^2_{V_{\phi_i(t)}}\mathcal{F}\big)(\mu_t)\, dt.$$

Now having these results in hand, we say that the stochastic process $\{\mu_t;\ t \geq 0\}$ solves the following SDE on $\mathbb{P}_{2,\infty}(M)$.

$$\circ d_t^I \mu_t = \sum_{i=0}^{N} V_{\phi_i(t)}(\mu_t) \circ dB_t^i, \quad \mu_0 = \rho\, dx.$$

Regular curves, parallel translations on $\mathbb{P}_{2,\infty}(M)$

J. Lott introduced the Levi-Civita covariant derivative $\bar\nabla_{V_{\psi_1}} V_{\psi_2}$ and obtained the expression:

$$\langle \bar\nabla_{V_{\psi_1}} V_{\psi_2}, V_{\psi_3}\rangle_{\bar{\mathbf{T}}_\mu} = \int_M \langle \nabla_{\nabla\psi_1}\nabla\psi_2, \nabla\psi_3\rangle\, \mu(dx).$$

Let

$$\Pi_\mu : L^2(M, TM; \mu) \to \bar{\mathbf{T}}_\mu$$

be the orthogonal projection; then

$$\big(\bar\nabla_{V_{\psi_1}} V_{\psi_2}\big)(\mu) = \Pi_\mu\big(\nabla_{\nabla\psi_1}\nabla\psi_2\big).$$

For $\mu \in \mathbb{P}_{2,\infty(M)}$ with $d\mu = \rho\, dx,\ \rho > 0$, we denote by Δ_μ the Witten Laplacian: $\Delta_\mu = \Delta + \langle \nabla \log\rho,\ \nabla\cdot\rangle$ and $\mathrm{div}_\mu(Z) = \mathrm{div}(Z) + \langle \nabla \log\rho, Z\rangle$, then

$$\Pi_\mu(Z) = \nabla\Delta_\mu^{-1}\big(\mathrm{div}_\mu(Z)\big).$$

Now let $\{c_t\}$ be a curve in $\mathbb{P}_{2,\infty}(M)$ defined by a flow of diffeomorphisms $X_{t,s}$ associate to ODE:

$$dX_{t,s} = \nabla\phi_t(X_{t,s})\, dt, \quad t \geq s, \quad X_s(x) = x,$$

with $c_t = (X_{t,0})_{\#}(\rho\, dx)$. Let $\{Y_t;\ t \in [0,1]\}$ be a family of vector fields along $\{c_t;\ t \in [0,1]\}$, that is, $Y_t \in \bar{\mathbf{T}}_{c_t}$. Suppose there are smooth functions $(t,x) \to \Phi_t(x)$ and $(t,x) \to \Psi_t(x)$ such that

$$\frac{d^I c_t}{dt} = V_{\Phi_t}, \quad Y_t = V_{\Psi_t}.$$

J. Lott obtained that if $\{Y_t;\ t \in [0,1]\}$ is parallel along $\{c_t;\ t \in [0,1]\}$, then $\{\nabla\Psi_t;\ t \in [0,1]\}$ is a solution to the following linear PDE (Lott equation)

$$\frac{d}{dt}\nabla\Psi_t + \Pi_{c_t}\Big(\nabla_{\nabla\Phi_t}\nabla\Psi_t\Big) = 0.$$

We can explicit the orthogonal projection Π_{c_t} in the case $M = \mathbb{T}$. A function v on $\mathbb{T}$ is the derivative of a function ϕ if and only if $\int_{\mathbb{T}} v(x)\,dx = 0$. The derivative of ϕ on $\mathbb{T}$ is denoted by $\partial_x \phi$. Let $\mu \in \mathbb{P}_{2,\infty}(\mathbb{T})$ with $\rho = \dfrac{d\mu}{dx} > 0$. Let $\partial_x \phi = \Pi_\mu(v)$; then for any function f,

$$\int_{\mathbb{T}} \partial_x f\, v(x)\rho(x)\,dx = \int_{\mathbb{T}} \partial_x f\, \partial_x \phi\, \rho(x)\,dx.$$

This implies that $\partial_x(v\rho) = \partial_x(\partial_x \phi\ \rho)$, so that for a constant K,

$$v\rho = \partial_x \phi\, \rho + K \quad \text{or} \quad v = \partial_x \phi + \frac{K}{\rho}.$$

Integrating the two hand sides over $\mathbb{T}$ yields $K = \dfrac{\int_{\mathbb{T}} v(x)dx}{\int_{\mathbb{T}} \frac{dx}{\rho}}$. Then

$$\Pi_\mu(v) = v - \frac{\int_{\mathbb{T}} v(x)dx}{\int_{\mathbb{T}} \frac{dx}{\rho}} \cdot \frac{1}{\rho}.$$

We put

$$\hat{\rho} = \frac{1}{\left(\int_{\mathbb{T}} \frac{dx}{\rho}\right) \rho}.$$

Note that $\int_{\mathbb{T}} \hat{\rho}\, dx = 1$. We will use Π_ρ instead of Π_μ. Then

$$\Pi_\rho(v) = v - \left(\int_{\mathbb{T}} v(x) dx\right)\, \hat{\rho}.$$

Let $\phi_t \in C^\infty(\mathbb{T})$ and (X_t) be the flow associated to $\dfrac{dX_t}{dt} = \partial_x \phi_t(X_t)$ and $c_t = (X_t)_\#(\rho dx)$. Set $\rho_t = \dfrac{dc_t}{dx}$ the density. Let $g_t \in C^2(\mathbb{T})$ such that $\int_{\mathbb{T}} g_t(x)\, dx = 0$. Then $\{g_t;\ t \in [0,1]\}$ is a solution to Lott equation if

$$\frac{dg_t}{dt} + \Pi_{\rho_t}\Big(\partial_x g_t\, \partial_x \phi_t\Big) = 0.$$

or

$$\frac{dg_t}{dt} = -\partial_x g_t\, \partial_x \phi_t + \left(\int_{\mathbb{T}} \partial_x g_t\, \partial_x \phi_t\, dx\right) \hat{\rho}_t.$$

Put $f_t = g_t(X_t)$. Then

$$\frac{df_t}{dt} = \Big(\int_{\mathbb{T}} \partial_x g_t \, \partial_x \phi_t \, dx \Big) \, \hat{\rho}_t(X_t).$$

Remark that

$$\int_{\mathbb{T}} \partial_x g_t \, \partial_x \phi_t \, dx = - \int_{\mathbb{T}} g_t \, \partial_x^2 \phi_t \, dx$$
$$= - \int_{\mathbb{T}} \frac{g_t \, \partial_x^2 \phi_t}{\rho_t} \, \rho_t \, dx = - \int_{\mathbb{T}} g_t(X_t) \Big(\frac{\partial_x^2 \phi_t}{\rho_t} \Big)(X_t) \, \rho \, dx.$$

Then f_t satisfies the following equation

$$\frac{df_t}{dt} = -\Big(\int_{\mathbb{T}} f_t \; \frac{\partial_x^2 \phi_t}{\rho_t}(X_t) \, \rho \, dx \Big) \, \hat{\rho}_t(X_t).$$

Define $\Lambda(t, f) = -\Big(\int_{\mathbb{T}} f \; \frac{\partial_x^2 \phi_t}{\rho_t}(X_t) \, \rho \, dx \Big) \, \hat{\rho}_t(X_t)$. Then

$$\frac{df_t}{dt} = \Lambda(t, f_t).$$

Lemma

There is a constant C_ϕ only dependent of ϕ such that

$$||\Lambda(t,f)-\Lambda(t,g)||_{L^2(\rho\,dx)} \leq C_\phi\,||f-g||_{L^2(\rho\,dx)}, \quad t\in[0,1].$$

Proof. Note that

$$\int_{\mathbb{T}}\Big(\frac{\partial_x^2\phi_t}{\rho_t}\Big)^2(X_t)\,\rho\,dx = \int_{\mathbb{T}}\frac{(\partial_x^2\phi_t)^2}{\rho_t}\,dx \leq ||\partial_x^2\phi_t||_\infty^2\int_{\mathbb{T}}\frac{dx}{\rho_t},$$

and $\int_{\mathbb{T}}\hat{\rho}_t(X_t)^2\,\rho\,dx = (\int_{\mathbb{T}}\frac{dx}{\rho_t})^{-1}$; it follows that

$$\int_{\mathbb{T}}\Big|\int_{\mathbb{T}} f\;\frac{\partial_x^2\phi_t}{\rho_t}(X_t)\,\rho\,dx\Big|^2\hat{\rho}_t(X_t)^2\,\rho\,dx \leq ||\partial_x^2\phi_t||_\infty\,||f||^2_{L^2(\rho\,dx)}$$

and global Lipschitz condition holds.

By classical theory of ODE, for $f_0 \in L^2(\rho\, dx)$, there is a unique solution f_t to above Equation. Set

$$g_t = f_t(X_t^{-1}).$$

We check that g_t is a solution to Lott Equation. Finally

Theorem

For any $g_0 \in \bar{\mathbf{T}}_{\rho dx}$ given, there is a unique solution $g_t \in \bar{\mathbf{T}}_{\rho_t dx}$ to parallel translation equation such that $\int_{\mathbb{T}} |g_t|^2\, \rho_t\, dx = \int_{\mathbb{T}} |g_0|^2 \rho\, dx$ for any $t \in [0, 1]$.

Stochastic parallel translations

Lott equation in determinist case

$$\frac{d}{dt}\nabla\Psi_t + \Pi_{c_t}\Big(\nabla_{\nabla\Phi_t}\nabla\Psi_t\Big) = 0,$$

becomes

$$\circ d_t(\nabla\Psi_t) = -\Pi_{\mu_t}\Big(\nabla_{\nabla\phi_i}\nabla\Psi_t\Big)\circ dB_t$$

where μ_t is a stochastic regular curve and B_t a Brownian motion. We will not discuss general situation, but only the case $\mathbb{P}_{2,\infty}(\mathbb{T})$.

Consider SDE on $\mathbb{T}$,

$$dX_t = \partial_x\phi_t(X_t)\circ dB_t.$$

Let $d\mu = \rho\, dx$ and $\mu_t = (X_t)_{\#}\mu$; set $\rho_t = \frac{d\mu_t}{dx}$. Suppose that $\{\partial_x\Psi_t;\ t\in[0,1]\}$ is a solution of parallel translations:

$$d_t\partial_x\Psi_t = -\Pi_{\rho_t}\big(\partial_x^2\Psi_t\ \partial_x\phi_t\big)\, dB_t + \Big(\frac{1}{2}R_t^{\Psi_t} - \frac{1}{2}S_t^{\Psi_t}\Big)\, dt.$$

Let $f_t = \partial\Psi_t(X_t)$. Then by Kunita-Itô-Wentzell formula, we get

$$
\begin{aligned}
d_t f_t = &-\Big(\int_{\mathbb{T}} \partial_x \Psi_t \, \partial_x^2 \phi_t \, dx\Big) \, \hat{\rho}_t(X_t) \, dB_t \\
&- \frac{1}{2}\Big(\int_{\mathbb{T}} \partial_x \Psi_t \, \partial_x^2 \phi_t \, dx\Big) \, (\partial_x^2 \phi_t)(X_t) \hat{\rho}_t(X_t) \, dt \\
&- \frac{1}{2}\Big(\int_{\mathbb{T}} \partial_x \Psi_t \, \partial_x\big(\partial_x^2 \phi_t \, \partial_x \phi_t\big) \, dx\Big) \hat{\rho}_t(X_t) \, dt \\
&+ \frac{3}{2}\Big(\int_{\mathbb{T}} \partial_x \Psi_t \, \partial_x^2 \phi_t \, dx\Big)\Big(\int_{\mathbb{T}} \partial_x^2 \phi_t \, \hat{\rho}_t \, dx\Big) \hat{\rho}_t(X_t) dt.
\end{aligned}
$$

We have

$$
\int_{\mathbb{T}} \partial_x \Psi_t \, \partial_x^2 \phi_t \, dx = \int_{\mathbb{T}} f_t \times \frac{\partial_x^2 \phi_t}{\rho_t}(X_t) \; \rho \, dx,
$$

$$
\int_{\mathbb{T}} \partial_x \Psi_t \, \partial_x\big(\partial_x^2 \phi_t \, \partial_x \phi_t\big) \, dx = \int_{\mathbb{T}} f_t \times \frac{\partial_x(\partial_x^2 \phi_t \, \partial_x \phi_t)}{\rho_t}(X_t) \; \rho \, dx.
$$

We introduce two notations

$$a_t = \frac{\partial_x^2 \phi_t}{\rho_t}(X_t), \quad b_t = \frac{\partial_x(\partial_x^2 \phi_t \, \partial_x \phi_t)}{\rho_t}(X_t).$$

Then $\{f_t; t \in [0,1]\}$ satisfies the following equation

$$\begin{aligned} d_t f_t = & - \Big(\int_{\mathbb{T}} f_t a_t \, \rho dx\Big) \hat{\rho}_t(X_t)\, dB_t - \frac{1}{2}\Big(\int_{\mathbb{T}} f_t a_t \, \rho dx\Big)\big(\hat{\rho}_t \, \partial_x^2 \phi_t\big)(X_t)\, dt \\ & - \frac{1}{2}\Big(\int_{\mathbb{T}} f_t b_t \, \rho dx\Big) \hat{\rho}_t(X_t)\, dt \\ & + \frac{3}{2}\Big(\int_{\mathbb{T}} f_t a_t \rho dx\Big)\Big(\int_{\mathbb{T}} \partial_x^2 \phi_t^2 \hat{\rho}_t \, dx\Big)\, \hat{\rho}_t(X_t)\, dt \\ & = \Lambda_1(t, f_t)\, dB_t + \Lambda_2(t, f_t)\, dt. \end{aligned}$$

As in above section,

$$\begin{aligned} & ||\Lambda_1(t,f) - \Lambda_1(t,g)||_{L^2(\rho dx)} + ||\Lambda_2(t,f) - \Lambda_2(t,g)||_{L^2(\rho dx)} \\ & \leq C_\phi \, ||f - g||_{L^2(\rho dx)}. \end{aligned}$$

By standard Picard iteration or by SDE on Hilbert spaces, there is a unique solution $\{f_t; t \in [0,1]\}$ to above Equation. Define $g_t = f_t(X_t^{-1})$. Remark that no SDE directly express X_t^{-1}.

Theorem

Suppose that $\int_{\mathbb{T}} g_0(x)dx = 0$, *then for any* $t \in [0,1]$, $\int_{\mathbb{T}} g_t(x)\,dx = 0$.

Proof. Let $\tilde{K}_t = \dfrac{d(X_t^{-1})_\#(dx)}{dx}$; then by Kunita,

$$\tilde{K}_t = \exp\Big(\int_0^t (\partial_x^2 \phi_s)(X_s) \circ dB_s\Big).$$

Note that $\int_{\mathbb{T}} g_t(x)\,dx = \int_{\mathbb{T}} f_t\,\tilde{K}_t\,dx$. We check that $\circ d_t \int_{\mathbb{T}} f_t \tilde{K}_t dx = 0$.

Combining all above results, finally we get

Theorem

Let $\partial_x \Psi_t = g_t$. Then for $\mu = \rho\, dx$ and $\mu_t = (X_t)_{\#}(\rho dx)$, $\{\partial_x \Psi_t;\ t \in [0,1]\}$ is the parallel translation along the stochastic regular curve $\{\mu_t;\ t \in [0,1]\}$, that is, $\partial_x \Psi_t \in \bar{\mathbf{T}}_{\mu_t}$ and

$$\int_{\mathbb{T}} |\partial_x \Psi_t|^2\, \mu_t(dx) = \int_{\mathbb{T}} |\partial_x \Psi_0|^2\, \rho dx, \quad t \in [0,1].$$

Brownian motion paths on $\mathbb{P}_{2,\infty}(\mathbb{T})$ and parallel translations along them.

Now let $\phi_{2k-1}(x) = \dfrac{\sin(kx)}{k}$ and $\phi_{2k}(x) = \dfrac{-\cos(kx)}{k}$. Consider a sequence of independent real BM $\{B_k; k \geq 1\}$.

For $N \geq 1$, let X_t^N be the flow associated to

$$dX_t^N = \sum_{k=1}^{N} \frac{1}{\alpha_k} \Big(\partial_x \phi_{2k-1}(X_t^N) \circ dB_{2k-1}(t) + \partial_x \phi_{2k}(X_t^N) \circ dB_{2k}(t) \Big),$$

$\mu_t^N = (X_t^N)_\#(\rho dx)$, and $\{\partial_x \Psi_t^N;\ t \in [0,1]\}$ parallel translation along $\{\mu_t^N; t \in [0,1]\}$. Then letting $N \to +\infty$,

$X_t^N \to X_t, \quad \mu_t^N \to \mu_t \quad \text{with } \mu_t = (X_t)_\#(\rho dx)$, and

$$\partial_x \Psi_t^N \sqrt{\rho_t^N} \quad \text{converges to} \quad \partial_x \Psi_t \sqrt{\rho_t} \quad \text{in } L^2(dx).$$

Theorem
$\{\partial_x \Psi_t;\ t \in [0,1]\}$ *is the parallel translation along the Brownian motion paths* $\{\mu_t;\ t \in [0,1]\}$*, that is,* $\partial_x \Psi_t \in \bar{\mathbf{T}}_{\mu_t}$ *and*

$$\int_{\mathbb{T}} |\partial_x \Psi_t|^2 \, \mu_t(dx) = \int_{\mathbb{T}} |\partial_x \Psi_0|^2 \, \rho dx, \quad t \in [0,1].$$